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A PROBLEM OF REISSNER–SAGOCI TYPE FOR A FINITE ELASTIC CYLINDER EMBEDDED IN AN ELASTIC LAYER

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Abstract—The problem considered is that of the torsion of a finite elastic cylinder which is embedded in an elastic medium of different shear modulus. By the use of integral transforms and the theory of dual integral equations, the problem is reduced to that of solving a Fredholm integral equation of the second kind. Numerical results have been displayed graphically.

1. INTRODUCTION

The problem of determining the torsional deformation of a semi-infinite, isotropic, homogeneous elastic solid when a circular cylinder is welded to its plane boundary and it is forced to rotate about its axis was first considered by Reissner and Sagoci (1944), Sagoci (1944) and Sneddon (1947). Sneddon (1966a) presented an integral transform solution for the torsional deformation of a half-space and of a circular cylinder with a rigidly clamped lateral surface. Bycroft (1956), Ufland (1959) and Gladwell (1969) solved the Reissner–Sagoci problem for an elastic layer of finite thickness when the layer face is either stress-free or rigidly clamped. Freeman and Keer (1967) investigated a torsion problem of an elastic cylinder bonded to an elastic half-space. And later Freeman and Keer (1970) extended their analysis to the torsion of a finite elastic rod which is partially bonded to a semi-infinite elastic cylinder of the same radius which in turn is embedded in an elastic half-space. Luco (1976) investigated the torsion problem of a rigid rod which is embedded in an elastic layer, the whole being perfectly bonded to a half-space with different shear modulus. Singh and Dhaliwal (1977) considered the Reissner–Sagoci problem for an elastic layer under torsion by a pair of circular discs on opposite faces. Dhaliwal *et al.* (1979) solved the Reissner–Sagoci problem for a semi-infinite elastic cylinder embedded in a half-space. Chebakov (1970) considered the Reissner–Sagoci problem for a finite cylinder with the curved surface and bottom face fixed. Karasudhi *et al.* (1984) considered Luco's problem under the assumption that the embedded rod was elastic rather than rigid. Recently Gladwell and Lameczyk (1989) solved the Reissner–Sagoci problem for a finite cylinder with stress-free curved sides and a fixed base.

In this paper we consider the problem of the torsion of a finite elastic cylinder which is embedded in an elastic layer with different shear modulus. It is assumed that the bottom flat surface of the cylinder and the surrounding layer is rigidly fixed.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

We are assuming that an elastic cylinder of radius a and shear modulus μ_1 is embedded in an elastic medium whose shear modulus is μ_2 as shown in Fig. 1. It is assumed that the cylinder is perfectly bonded to the surrounding elastic layer and a torque is applied to the cylinder, through a rigid disc of radius c, which is bonded to its top flat surface. In terms of cylindrical polar coordinates (r, θ, z), the displacement field and corresponding non-zero stress components are given by

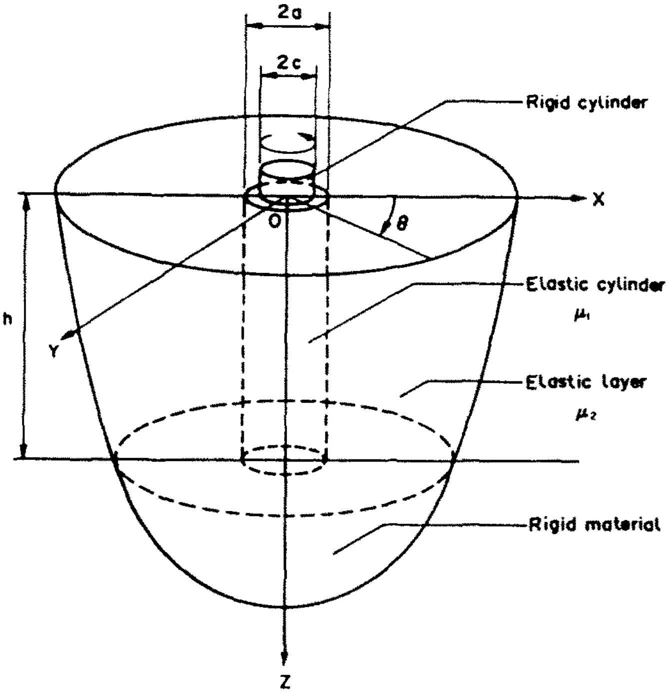

Fig. 1. Torsion of an elastic cylinder bonded to a dissimilar elastic layer.

$$u = 0, \quad v = v(r, z), \quad w = 0, \quad \sigma_{\theta z}(r, z) = \mu \frac{\partial v}{\partial z}, \quad \sigma_{r\theta}(r, z) = \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right), \tag{1}$$

and the equation of equilibrium is given as follows:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0. \tag{2}$$

Using the method of separation of variables, it is easy to show that the following are basic solutions of eqn (2) for $v(r, z)$:

(I) $J_1(\xi r) \exp(\pm \xi z)$, (II) $Y_1(\xi r) \exp(\pm \xi z)$,

(III) $I_1(\xi r) \cos(\xi z)$ or $I_1(\xi r) \sin(\xi z)$,

(IV) $K_1(\xi r) \cos(\xi z)$ or $K_1(\xi r) \sin(\xi z)$, (V) $rz, r, 1/r, z/r$;

where J_ν, Y_ν are Bessel functions of the first and second kind and of order ν respectively; I_ν, K_ν are modified Bessel functions of the first and second kind respectively and of order ν and ξ is a real parameter.

We assume that the rigid disc bonded to the cylinder is turned through a small angle ε and that the height of the cylinder and the layer is h. We therefore consider the problem of determining the stress and displacement field in the cylinder and the layer with the following boundary and continuity conditions:

$$v(r, 0) = \varepsilon r, \quad 0 \leqslant r < c, \tag{3}$$

$$\sigma_{\theta z}(r, 0) = 0, \quad c \leqslant r < a, \tag{4}$$

$$\bar{\sigma}_{\theta z}(r, 0) = 0, \quad r > a, \tag{5}$$

$$v(r,b) = 0, \quad r < a, \tag{6}$$

$$\hat{v}(r,b) = 0, \quad r > a, \tag{7}$$

$$v(a,z) = \hat{v}(a,z), \quad 0 \leqslant z \leqslant b, \tag{8}$$

$$\sigma_{r\theta}(a,z) = \hat{\sigma}_{r\theta}(a,z), \quad 0 \leqslant z \leqslant b, \tag{9}$$

where v, $\sigma_{\theta z}$ and $\sigma_{r\theta}$ are the non-zero displacement and stress components in the cylinder while $\hat{v}$, $\hat{\sigma}_{\theta z}$ and $\hat{\sigma}_{r\theta}$ are their counterparts in the layer.

In the solution of this problem we shall make use of the following notations from Sneddon (1951): the Fourier operators $\mathscr{F}_s$ and $\mathscr{F}_c$ are defined by the equations

$$\mathscr{F}_s[f(z)\,;\, z \to \xi] = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(z) \sin(\xi z)\,\mathrm{d}z,$$

$$\mathscr{F}_c[f(z)\,;\, z \to \xi] = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(z) \cos(\xi z)\,\mathrm{d}z,$$

the Hankel operator $\mathscr{H}_v$ is defined by the equation

$$\mathscr{H}_v[f(\xi)\,;\, \xi \to z] = \int_0^\infty \xi f(\xi) J_v(\xi r)\,\mathrm{d}\xi,$$

and the Abel operator of the first kind, $\mathscr{A}_1$, is defined by the equation

$$\mathscr{A}_1[f(t)\,;\, t \to r] = \left(\frac{2}{\pi}\right)^{1/2} \int_0^r f(t)[r^2 - t^2]^{-1/2}\,\mathrm{d}t,$$

then we have the following properties of these operators

$$\mathscr{F}_s^{-1} = \mathscr{F}_s, \quad \mathscr{F}_c^{-1} = \mathscr{F}_c, \quad \mathscr{H}_v^{-1} = \mathscr{H}_v, \quad \mathscr{A}_1^{-1}[f(r)\,;\, t] = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{A}_1[rf(r)\,;\, t].$$

3. DERIVATION OF THE DUAL INTEGRAL EQUATIONS

In this section we introduce a combination of the basic solutions for $v(r,z)$ obtained in Section 2, by means of this combination we are able to reduce the problem of solving the mixed boundary problem to that of a pair of dual integral equations.

For $0 \leqslant r < a$, we may assume the following representation:

$$v(r,z) = a_0 r(b-z) + \mathscr{H}_1[\xi^{-1} A(\xi) \sinh[\xi(b-z)]\,;\, \xi \to r] + \sum_{n=1}^{\infty} \xi_n^{-1} B_n \cos(\xi_n z) I_1(\xi_n r), \tag{10}$$

and for $r > a$ we may assume that

$$\hat{v}(r,z) = \sum_{n=1}^{\infty} \xi_n^{-1} C_n \cos(\xi_n z) K_1(\xi_n r), \tag{11}$$

where $a_0 = \varepsilon/b$, while A, B_n, C_n and ξ_n are to be determined later.

From eqns (1) we have

$$\sigma_{r\theta}(r,z) = -\mu_1\mathscr{H}_2[A(\xi)\sinh[\xi(b-z)];\xi\to r]+\mu_1\sum_{n=1}^{\infty}B_n\cos(\xi_n z)I_2(\xi_n r), \tag{12}$$

$$\hat{\sigma}_{r\theta}(r,z) = -\mu_2\sum_{n=1}^{\infty}C_n\cos(\xi_n z)K_2(\xi_n r), \tag{13}$$

$$\sigma_{\theta z}(r,z) = -\mu_1\mathscr{H}_1[A(\xi)\cosh[\xi(b-z)];\xi\to r]-\mu_1\sum_{n=1}^{\infty}B_n\sin(\xi_n z)I_1(\xi_n r)-a_0\mu_1 r, \tag{14}$$

$$\hat{\sigma}_{\theta z}(r,z) = -\mu_2\sum_{n=1}^{\infty}C_n\sin(\xi_n z)K_1(\xi_n r). \tag{15}$$

The conditions (6) and (7) may be satisfied by taking

$$\cos(\xi_n b) = 0,$$

which gives

$$\xi_n = (2n-1)\pi/2b, \quad n = 1,2,3,\ldots \tag{16}$$

The condition (8) yields

$$a_0 a(b-z)+\mathscr{H}_1[\xi^{-1}A(\xi)\sinh[\xi(b-z)];\xi\to a] = \sum_{n=1}^{\infty}\xi_n^{-1}[C_nK_1(\xi_n a)-B_nI_1(\xi_n a)]\cos(\xi_n z), \tag{17}$$

and the condition (9) yields

$$\mathscr{H}_2[A(\xi)\sinh[\xi(b-z)];\xi\to a] = \sum_{n=1}^{\infty}[B_nI_2(\xi_n a)+\bar{\mu}C_nK_2(\xi_n a)]\cos(\xi_n z), \tag{18}$$

where $\bar{\mu} = \mu_2/\mu_1$.

Since $\{\cos(\xi_n z)\}_{n=1,2,3,\ldots}$ are orthogonal over the interval $(0,b)$ and

$$\int_0^b \sinh[\xi(b-z)]\cos(\xi_n z)\,dz = \frac{\xi\cosh(\xi b)}{(\xi^2+\xi_n^2)}, \tag{19}$$

$$\int_0^b (b-z)\cos(\xi_n z)\,dz = \frac{1}{\xi_n^2}, \tag{20}$$

eqns (17) and (18) lead to the following equations

$$-B_nI_1(\xi_n a)+C_nK_1(\xi_n a) = \frac{2\xi_n}{b}\left\{\int_0^\infty \frac{\xi A(\xi)\cosh(\xi b)J_1(\xi a)}{\xi^2+\xi_n^2}\,d\xi+\frac{a_0 a}{\xi_n^2}\right\} = G_1(n), \tag{21}$$

$$B_nI_2(\xi_n a)+\bar{\mu}C_nK_2(\xi_n a) = \frac{2}{b}\int_0^\infty \frac{\xi^2 A(\xi)\cosh(\xi b)J_2(\xi a)}{\xi^2+\xi_n^2}\,d\xi = G_2(n). \tag{22}$$

Solving eqns (21) and (22) for B_n we obtain

$$B_n = [G_2(n)K_1(\xi_n a) - \bar{\mu}G_1(n)K_2(\xi_n a)]/\Delta(n), \tag{23}$$

where

$$\Delta(n) = I_2(\xi_n a)K_1(\xi_n a) + \bar{\mu}I_1(\xi_n a)K_2(\xi_n a). \tag{24}$$

From eqn (15) we find that the condition (5) is identically satisfied and boundary conditions (3) and (4) will be satisfied if $A(\xi)$ is the solution of the following dual integral equations:

$$\mathscr{H}_1[\xi^{-1}A(\xi)\sinh(\xi b);\ \xi \to r] + \sum_{n=1}^{\infty} \xi_n^{-1}B_n I_1(\xi_n r) = 0, \quad r < c, \tag{25}$$

$$\mathscr{H}_1[A(\xi)\cosh(\xi b);\ \xi \to r] = -a_0 r, \quad c < r < a. \tag{26}$$

4. REDUCTION TO INTEGRAL EQUATION OF FREDHOLM TYPE

We can reduce the problem of solving the dual integral equations (25) and (26) to that of solving an integral equation of Fredholm type of the second kind by means of an integral representation for $A(\xi)$, which identically satisfies eqn (26).

If we take

$$\phi_0(t) = -\left(\frac{2}{\pi}\right)^{1/2}\begin{cases} t[(a^2-t^2)^{1/2}-(c^2-t^2)^{1/2}], & t < c, \\ t(a^2-t^2)^{1/2}, & c < t < a, \\ 0, & t > a, \end{cases} \tag{27}$$

we have

$$\mathscr{F}_s[\phi_0(t);\ t \to \xi] = -\int_c^a r^2 J_1(\xi r)\,\mathrm{d}r, \tag{28}$$

$$\mathscr{H}_1\{\mathscr{F}_s[\phi_0(t);\ t \to \xi];\ \xi \to r\} = \begin{cases} 0, & 0 \leqslant r < c, \\ -r, & c < r < a, \\ 0, & r > a. \end{cases} \tag{29}$$

It is easy to show by using Sneddon (1966b) that if we take

$$A(\xi) = \frac{a_0}{\cosh(\xi b)}\mathscr{F}_s[\phi(t)+\phi_0(t);\ t \to \xi], \tag{30}$$

where $\phi(t)$ is a new unknown function defined in $(0, \infty)$ such that $\phi(t) = 0$ for $t > c$, then eqn (26) will be satisfied identically.

Substituting eqn (30) into eqns (21) and (22) and using the following integrals given by Erdelyi (1954):

$$\int_0^\infty \frac{\xi^2 \sin(\xi t)J_2(\xi a)\,\mathrm{d}\xi}{\xi^2+\xi_n^2} = \xi_n \sinh(\xi_n t)K_2(\xi_n a), \quad t < a, \tag{31}$$

$$\int_0^\infty \frac{\xi \sin(\xi t)J_1(\xi a)\,\mathrm{d}\xi}{\xi^2+\xi_n^2} = \sinh(\xi_n t)K_1(\xi_n a), \quad t < a, \tag{32}$$

we obtain

$$G_1(n) = g_{11}(n)+g_{12}(n)+g_{13}(n), \quad G_2(n) = g_{21}(n)+g_{22}(n), \tag{33}$$

where

$$g_{11}(n) = \frac{2a_0\xi_n}{b}\left(\frac{2}{\pi}\right)^{1/2}\int_0^c \phi(t)\sinh(\xi_n t)K_1(\xi_n a)\,\mathrm{d}t,$$

$$g_{12}(n) = \frac{2a_0\xi_n}{b}\left(\frac{2}{\pi}\right)^{1/2}\int_0^a \phi_0(t)\sinh(\xi_n t)K_1(\xi_n a)\,\mathrm{d}t,$$

$$g_{13}(n) = \frac{2a_0 a}{b\xi_n},$$

$$g_{21}(n) = \frac{2a_0\xi_n}{b}\left(\frac{2}{\pi}\right)^{1/2}\int_0^c \phi(t)\sinh(\xi_n t)K_2(\xi_n a)\,\mathrm{d}t,$$

$$g_{22}(n) = \frac{2a_0\xi_n}{b}\left(\frac{2}{\pi}\right)^{1/2}\int_0^a \phi_0(t)\sinh(\xi_n t)K_2(\xi_n a)\,\mathrm{d}t, \tag{34}$$

and hence from eqn (23), we obtain that

$$B_n = \{[g_{21}(n)K_1(\xi_n a)-\bar{\mu}g_{11}(n)K_2(\xi_n a)] + [g_{22}(n)K_1(\xi_n a)-\bar{\mu}g_{12}(n)K_2(\xi_n a)]-\bar{\mu}g_{13}(n)K_2(\xi_n a)\}/\Delta(n). \tag{35}$$

Operating on eqn (25) by $x^{-1}\mathscr{A}_1^{-1}[r;x]$ and using the following results:

$$x^{-1}\mathscr{A}_1^{-1}\{r\mathscr{H}_1[\xi^{-1}A(\xi);\xi\to r];r\to\xi\} = \mathscr{F}_s[A(\xi);\xi\to x], \tag{36}$$

$$x^{-1}\mathscr{A}_1^{-1}[rI_1(\xi_n r);r\to x] = \left(\frac{2}{\pi}\right)^{1/2}\sinh(\xi_n x), \tag{37}$$

we obtain

$$\mathscr{F}_s[A(\xi)\sinh(\xi b);\xi\to x]+\left(\frac{2}{\pi}\right)^{1/2}\sum_{n=1}^{\infty}\xi_n^{-1}B_n\sinh(\xi_n x) = 0, \quad 0\leqslant x<c. \tag{38}$$

Using expressions (30) and (35), we get

$$\mathscr{F}_s[A(\xi)\sinh(\xi b);\xi\to x] = a_0\phi(x)-\frac{4a_0}{\pi}\int_0^c\phi(t)\,\mathrm{d}t\int_0^\infty(1+\mathrm{e}^{2\xi b})^{-1}\sin(\xi t)\sin(\xi x)\,\mathrm{d}\xi + a_0\phi_0(x)-\frac{4a_0}{\pi}\int_0^a\phi_0(t)\,\mathrm{d}t\int_0^\infty(1+\mathrm{e}^{2\xi b})^{-1}\sin(\xi t)\sin(\xi x)\,\mathrm{d}\xi, \tag{39}$$

and

$$\left(\frac{2}{\pi}\right)^{1/2}\sum_{n=1}^{\infty}\xi_n^{-1}B_n\sinh(\xi_n x) = a_0\int_0^c\phi(t)\left[\sum_{n=1}^{\infty}Q(n)\sinh(\xi_n t)\sinh(\xi_n x)\right]\mathrm{d}t + a_0\int_0^a\phi_0(t)\left[\sum_{n=1}^{\infty}Q(n)\sinh(\xi_n t)\sinh(\xi_n x)\right]\mathrm{d}t+a_0R(x), \tag{40}$$

where

$$Q(n) = \frac{4}{\pi b \Delta(n)}(1-\bar{\mu})K_1(\xi_n a)K_2(\xi_n a),$$

$$R(x) = -\frac{2a\bar{\mu}}{b}\left(\frac{2}{\pi}\right)^{1/2}\sum_{n=1}^{\infty}\frac{K_2(\xi_n a)\sinh(\xi_n z)}{\Delta(n)\xi_n^2}. \tag{41}$$

Substituting from (39) and (40) into eqn (38) we find that $\phi(t)$ must satisfy the following integral equation:

$$\phi(x) - \int_0^c \phi(t)M(x,t)\,\mathrm{d}t - \int_0^c \phi(t)N(x,t)\,\mathrm{d}t = -f(x), \quad 0 \leqslant x \leqslant c, \tag{42}$$

where

$$f(x) = \phi_0(x) - \int_0^a \phi_0(t)M(x,t)\,\mathrm{d}t - \int_0^a \phi_0(t)N(x,t)\,\mathrm{d}t + R(x), \tag{43}$$

$$M(x,t) = \frac{4}{\pi}\int_0^\infty (1+\mathrm{e}^{2\xi b})^{-1}\sin(\xi t)\sin(\xi x)\,\mathrm{d}\xi, \tag{44}$$

$$N(x,t) = \sum_{n=1}^{\infty} Q(n)\sinh(\xi_n t)\sinh(\xi_n x). \tag{45}$$

For large n,

$$Q(n)\sinh(\xi_n t)\sinh(\xi_n x) = O(\exp[-\xi_n(2a-x-t)]),$$

and hence the convergence of the series in eqn (45) is fast.

To calculate the torque T necessary to produce the prescribed rotation, of the rigid disc bonded to the cylinder, as given by condition (3) we need to calculate the expression for $\sigma_{\theta z}$ at $z = 0$. Now from eqns (14) and (30), we get

$$\begin{aligned}\sigma_{\theta z}(r,0) &= -\mu_1\mathscr{H}_1[A(\xi)\cosh(\xi b);\, \xi \to r] - a_0\mu_1 r \\ &= -a_0\mu_1\mathscr{H}_1\{\mathscr{F}_s[\phi(t);\, t\to\xi];\, \xi\to r\} - a_0\mu_1\mathscr{H}_1\{\mathscr{F}_s[\phi_0(t);\, t\to\xi];\, \xi\to r\} - a_0\mu_1 r, \\ &\qquad r < c,\end{aligned} \tag{46}$$

and we find that

$$\int_0^c r^2\mathscr{H}_1\{\mathscr{F}_s[\phi_0(t);\, t\to\xi];\, \xi\to r\}\,\mathrm{d}r = 0, \tag{47}$$

$$\mathscr{H}_1\{\mathscr{F}_s[\phi(t);\, t\to\xi];\, \xi\to r\} = -\frac{\mathrm{d}}{\mathrm{d}r}\mathscr{H}_0\{\xi^{-1}\mathscr{F}_s[\phi(t);\, t\to\xi];\, \xi\to r\}. \tag{48}$$

The expression for T is given by

$$T = -2\pi\int_0^c r^2\sigma_{\theta z}(r,0)\,\mathrm{d}r. \tag{49}$$

Using (46), (47) and (48) in (49), we find that

$$T = \frac{\pi\mu_1\varepsilon c^4}{2b} + 4(2\pi)^{1/2}\frac{\mu_1\varepsilon}{b}\int_0^c t\phi(t)\,dt. \tag{50}$$

5. PARTICULAR CASES

Case (*a*) $\mu_2 \to \infty$

Letting $\mu_2 \to \infty$ in the results of the previous section, we get the results for the case in which the elastic cylinder is embedded in a rigid medium as a limiting case. Since

$$\frac{1-\bar{\mu}}{\Delta(n)} \to \frac{-1}{I_1(\xi_n a)K_2(\xi_n a)}, \quad \text{as} \quad \mu_2 \to \infty,$$

we obtain the expression for the kernel

$$N(x,t) = \frac{4}{\pi b}\sum_{n=1}^{\infty}\frac{K_1(\xi_n a)}{I_1(\xi_n a)}\sinh(\xi_n t)\sinh(\xi_n x),$$

and

$$R(x) = -\frac{2a}{b}\left(\frac{2}{\pi}\right)^{1/2}\sum_{n=1}^{\infty}\frac{\sinh(\xi_n x)}{I_1(\xi_n a)\xi_n^2},$$

for $\mu_2 \to \infty$. With these modifications the solution for this case is given by eqns (42) and (50).

Case (*b*) $\mu_2 \to 0$

In this case, if we let $\mu_2 \to 0$, we get the solution for the case in which the elastic cylinder is free of stress on its curved surface. And

$$\frac{1-\bar{\mu}}{\Delta(n)} \to \frac{1}{I_2(\xi_n a)K_1(\xi_n a)}, \quad \text{as} \quad \mu_2 \to 0,$$

hence the kernel $N(x,t)$ becomes

$$N(x,t) = \frac{4}{\pi b}\sum_{n=1}^{\infty}\frac{K_2(\xi_n a)}{I_2(\xi_n a)}\sinh(\xi_n t)\sinh(\xi_n x),$$

and $R(x) = 0$, as $\mu_2 \to 0$ and these results are in agreement with Gladwell and Lemczyk (1989).

6. NUMERICAL RESULTS AND CONCLUSIONS

Numerical values of $\phi(x)$ for $x = (0.0, 0.1, 0.2, \ldots, 1.0)c$ have been calculated from the integral eqn (42) by reducing it to algebraic equations and hence the numerical values of the dimensionless ratio of torque T/T_0 have been calculated from eqn (50), where $T_0 = 16\mu_1\varepsilon c^3/3$ is the torque for the corresponding Reissner–Sagoci problem for the semi-infinite space.

Numerical values of T/T_0 have been calculated for the following values of b/c, a/c and $\bar{\mu} = \mu_2/\mu_1$:

$$b/c = 0.2(0.1)0.5, 1.0, 2.0, 10.0; \quad a/c = 1.0(0.2)2.0, 3.0(1.0)10.0;$$

$$\bar{\mu} = 0.0, 0.5, 2.0; \text{ and } \quad \bar{\mu} \to \infty,$$

and these have been displayed in Figs 2–5.

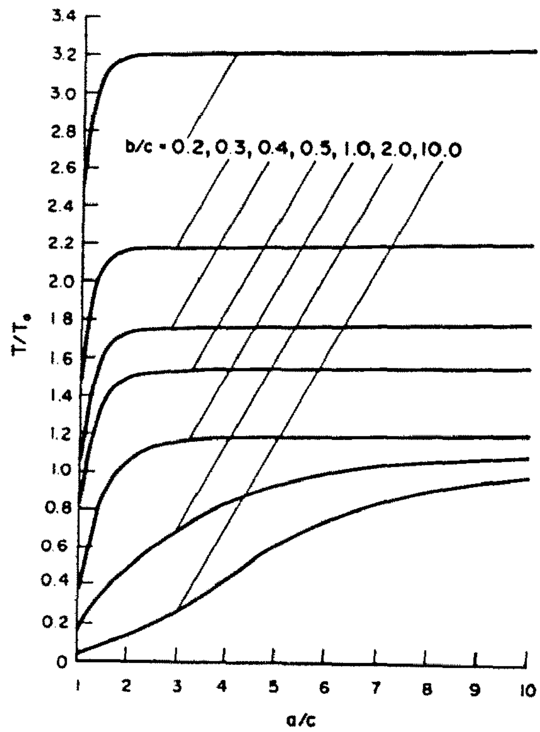

Fig. 2. Values of T/T_0 displayed against a/c for $b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0$ and $\bar{\mu} = 0.0$.

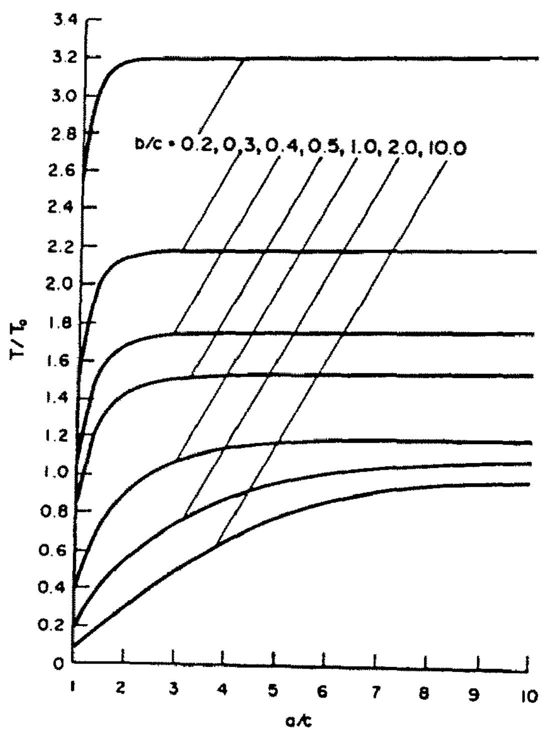

Fig. 3. Values of T/T_0 displayed against a/c for $b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0$ and $\bar{\mu} = 0.5$.

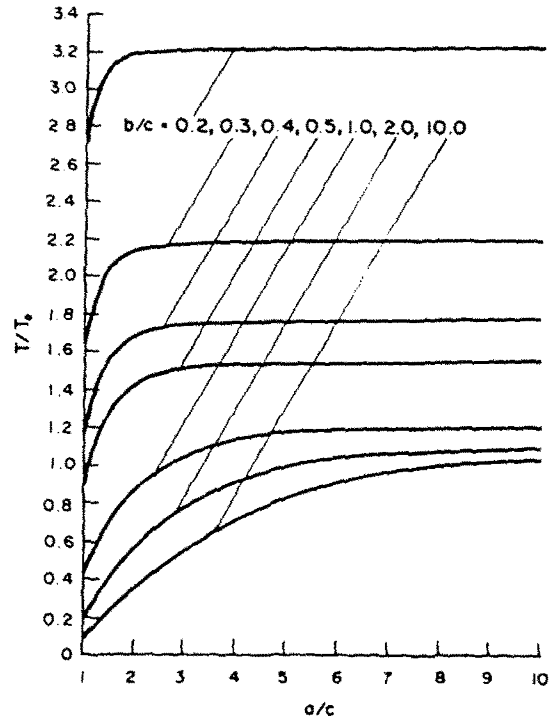

Fig. 4. Values of T/T_0 displayed against a/c for b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0 and $\bar{\mu}$ = 2.0.

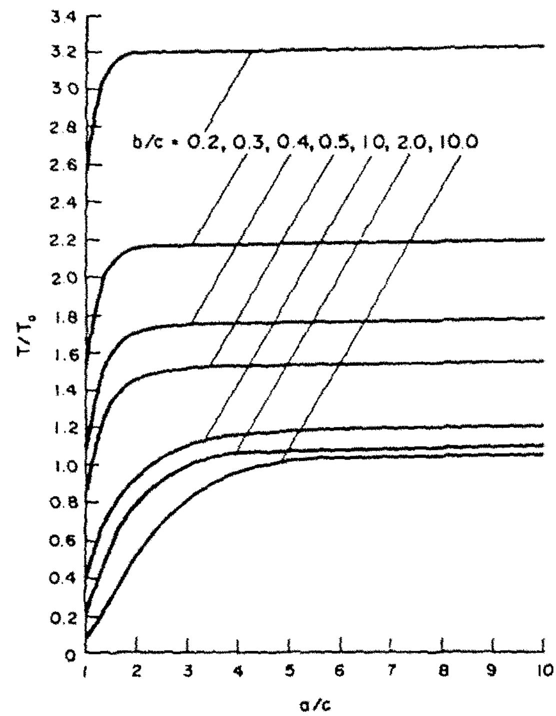

Fig. 5. Values of T/T_0 displayed against a/c for b/c = 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, 10.0 and $\bar{\mu} \to 0$.

From the figures, we observe that the torque T decreases as the height b of the cylinder increases and the torque increases as the radius a of the cylinder increases. We also observe that the ratio T/T_0 approaches to 1 with the simultaneous increase of a/c and b/c to infinity.

Table 1. Numerical values of T/T_0 for $\beta_1 = 0$, $\beta_2 = 0$ (Karashudhi *et al.*) $a/c = 1$, $\bar{\mu} \to \infty$ (He and Dhaliwal)

$h/a = b/a$*	1.0	2.0	10.0	
Karashudhi *et al.*	0.80	0.59	0.14	
	0.69	0.52	0.12	Quadratic
	0.62	0.45	0.10	Exponential
				Linear
He and Dhaliwal	0.41	0.21	0.09	

*The thickness of the elastic layer is taken as h in Karashudhi *et al.* and as b in He and Dhaliwal and a is the radius of the elastic cylinder.

We will now compare our numerical results with those of Karashudhi *et al.* (1984). We extrapolate their numerical values of T/T_0 given in Table 5 for $\beta_1 = 0$; $\beta_2 = 1.0, 0.75, 0.50, 0.25$ and find the values of T/T_0 for $\beta_1 = 0$, $\beta_2 = 0$. The results are obtained by using three different methods of extrapolation, namely linear, quadratic and exponential approximations. Our corresponding numerical values of T/T_0 are obtained for $a/c = 1.0$ and $\bar{\mu} \to \infty$. The numerical values of T/T_0 are given in Table 1.

We notice from the comparison of the above numerical values that when the layer thickness is ten times larger than the radius of the elastic cylinder, our results give only marginally lower values of T/T_0 as compared to theirs. But when the ratio of $h/a = b/a$ is smaller, the values of T/T_0 are considerably lower in our case, which shows the effect of our three-dimensional treatment of the problem.

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